

**International Journal of Pure and Applied Mathematics****Volume 109 No. 4 2016, 763-777**

ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)

url: <http://www.ijpam.eu>

doi: 10.12732/ijpam.v109i4.1

**DIRECT SOLUTION OF FOURTH ORDER ORDINARY  
DIFFERENTIAL EQUATIONS USING A ONE STEP  
HYBRID BLOCK METHOD OF ORDER FIVE**Zurni Omar<sup>1</sup>, Ra'ft Abdelrahim<sup>2 §</sup><sup>1,2</sup>Department of Mathematics

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**Abstract:** In this article, a power series of order eight is adopted as a basis function to develop one step hybrid block method with three off step points for solving general fourth order ordinary differential equations. The strategy is employed for the developing this method are interpolating the power series at  $x_n$  and all off-step points and collocating its fourth derivative at all points in the selected interval. The method derived is proven to be consistent, zero stable and convergent with order five. Taylors series is used to supply the starting values for the implementation of the method while the performance of the method is tasted by solving linear and non-linear problems.

**AMS Subject Classification:** 65L05, 65L06, 65L20

**Key Words:** hybrid method, block method, fourth order differential equation, power series, three off step points

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Received: February 20, 2016

Revised: August 11, 2016

Published: October 4, 2016

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url: [www.acadpubl.eu](http://www.acadpubl.eu)

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## 1. Introduction

Ordinary differential equations are frequently used in mathematical modelling in order to analyse and understand variety of real world problems. Hence, finding the solution to these equations is very crucial. In this study, we are interested in finding the numerical solution of the general fourth order initial value problem of the form

$$\begin{aligned} y^{iv} &= f(x, y, y', y'', y'''), \quad x \in [a, b], \\ y(a) &= \eta_0, \quad y'(a) = \eta_1, \quad y''(a) = \eta_2, \quad y'''(a) = \eta_3. \end{aligned} \quad (1)$$

Equation (1) can be solved by reducing it into an equivalent system of four first-order IVPs equations and then several suitable numerical methods for solving this systems are employed. However, [11] had shown that replacing this approach by direct method will reduce computational work. Since then, direct methods were widely used for solving IVPs (see [3], [5], [6] and [8]).

Recently, scholars have proposed a hybrid block methods for solving equation (1) directly in order to avoid the shortcoming in reduction method and to take the advantages of hybrid and block methods which includes overcoming zero stability barrier and generating numerical solutions simultaneously [7]. Some work in hybrid block methods can be found in [1], [2], [9], [10], and [4].

## 2. Methodology

In deriving the new method, the following power series is used as an approximate solution to (2). Power series as below is used for the derivation the new method

$$y(x) = \sum_{i=0}^{d+c-1} a_i \left( \frac{x - x_n}{h} \right)^i, \quad n = 0, 1, 2, \dots, N-1, \quad x \in [x_n, x_{n+1}], \quad (2)$$

where  $d = 4$  and  $c = 5$  denote the number of interpolation and collocation points respectively,  $h = x_n - x_{n-1}$  and  $a = x_0 < x_1 < \dots < x_{N-1} < x_N = b$ .

Interpolating (2) at  $x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}$  and collocating the fourth derivative of (2) at all points in the interval i.e at  $x_n, x_{n+\frac{1}{4}}, x_{n+\frac{1}{3}}, x_{n+\frac{2}{3}}$  and  $x_{n+1}$  produce a system of equations which can be written in matrix form

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} & \frac{1}{1024} & \frac{1}{4096} & \frac{1}{16384} & \frac{1}{65536} \\
1 & \frac{1}{3} & \frac{1}{9} & \frac{1}{27} & \frac{1}{81} & \frac{1}{243} & \frac{1}{729} & \frac{1}{2187} & \frac{1}{6561} \\
1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{128} & \frac{1}{256} \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{30}{h^4} & \frac{45}{(2h^4)} & \frac{105}{(8h^4)} & \frac{105}{(16h^4)} \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{40}{h^4} & \frac{40}{h^4} & \frac{280}{(9h^4)} & \frac{560}{(27h^4)} \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{80}{h^4} & \frac{160}{h^4} & \frac{2240}{(9h^4)} & \frac{8960}{(27h^4)} \\
0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120}{h^4} & \frac{360}{h^4} & \frac{840}{h^4} & \frac{1680}{h^4}
\end{pmatrix}
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5 \\
a_6 \\
a_7 \\
a_8
\end{pmatrix}
=
\begin{pmatrix}
y_n \\
y_{n+\frac{1}{4}} \\
y_{n+\frac{1}{3}} \\
y_{n+\frac{2}{3}} \\
f_n \\
f_{n+\frac{1}{4}} \\
f_{n+\frac{1}{3}} \\
f_{n+\frac{2}{3}} \\
f_{n+1}
\end{pmatrix} \quad (3)$$

Using Gaussian elimination method, the values of  $a_i$ 's,  $i = 0(1)8$  in equation (2) are obtained. The values of  $a_i$ 's are then substituted into equation (2) to yield a continuous implicit scheme of the form

$$y(x) = \sum_{i=0, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}} \alpha_i(x) y_{n+i} + \sum_{i=0}^1 \beta_i(x) f_{n+i} + \sum_{i=\frac{1}{4}, \frac{1}{3}, \frac{2}{3}} \beta_i(x) f_{n+i} \quad (4)$$

Differentiating equation (4) once, twice and thrice, we obtain

$$\begin{aligned}
y'(x) &= \sum_{i=0, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}} \frac{d}{dx} \alpha_i(x) y_{n+i} + \sum_{i=0}^1 \frac{d}{dx} \beta_i(x) f_{n+i} \\
&+ \sum_{i=\frac{1}{4}, \frac{1}{3}, \frac{2}{3}} \frac{d}{dx} \beta_i(x) f_{n+i}, \quad (5)
\end{aligned}$$

$$\begin{aligned}
y''(x) &= \sum_{i=0, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}} \frac{d^2}{dx^2} \alpha_i(x) y_{n+i} + \sum_{i=0}^1 \frac{d^2}{dx^2} \beta_i(x) f_{n+i} \\
&+ \sum_{i=\frac{1}{4}, \frac{1}{3}, \frac{2}{3}} \frac{d^2}{dx^2} \beta_i(x) f_{n+i}, \quad (6)
\end{aligned}$$

$$\begin{aligned}
y'''(x) &= \sum_{i=0, \frac{1}{4}, \frac{1}{3}, \frac{2}{3}} \frac{d^3}{dx^3} \alpha_i(x) y_{n+i} + \sum_{i=0}^1 \frac{d^3}{dx^3} \beta_i(x) f_{n+i} \\
&+ \sum_{i=\frac{1}{4}, \frac{1}{3}, \frac{2}{3}} \frac{d^3}{dx^3} \beta_i(x) f_{n+i}, \quad (7)
\end{aligned}$$

where

$$\alpha_0 = \frac{((h - 3x + 3x_n)(h - 4x + 4x_n)(2h - 3x + 3x_n))}{(2h^3)}$$

$$\alpha_{\frac{1}{4}} = \frac{(64(x - x_n)(h - 3x + 3x_n)(2h - 3x + 3x_n))}{(5h^3)}$$

$$\alpha_{\frac{1}{3}} = \frac{-(9(x - x_n)(h - 4x + 4x_n)(2h - 3x + 3x_n))}{h^3}$$

$$\alpha_{\frac{2}{3}} = \frac{(9(x - x_n)(h - 3x + 3x_n)(h - 4x + 4x_n))}{(10h^3)}$$

$$\begin{aligned} \beta_0 = & \frac{((x - x_n)^4)}{24} + \frac{(471353h^2(x - x_n)^2)}{209018880} - \frac{(19(x - x_n)^5)}{(240h)} \\ & + \frac{(31(x - x_n)^6)}{(360h^2)} - \frac{(27(x - x_n)^7)}{(560h^3)} + \frac{(3(x - x_n)^8)}{(280h^4)} \\ & - \frac{(14381h(x - x_n)^3)}{1105920} - \frac{(17329h^3(x - x_n))}{104509440} \end{aligned}$$

$$\begin{aligned} \beta_{\frac{1}{4}} = & \frac{-((352(x - x_n)^6))}{(675h^2)} - \frac{(64(x - x_n)^5)}{(225h)} - \frac{(476041h^2(x - x_n)^2)}{24494400} \\ & + \frac{(16(x - x_n)^8)}{(175h^4)} + \frac{(7357h(x - x_n)^3)}{129600} + \frac{(24449h^3(x - x_n))}{12247200} \\ & - \frac{(64(x - x_n)^7)}{(175h^3)} \end{aligned}$$

$$\begin{aligned} \beta_{\frac{1}{3}} = & \frac{-((1507h^2(x - x_n)^2))}{516096} + \frac{(9(x - x_n)^5)}{(40h)} - \frac{(39(x - x_n)^6)}{(80h^2)} \\ & + \frac{(207(x - x_n)^7)}{(560h^3)} - \frac{(27(x - x_n)^8)}{(280h^4)} - \frac{(7513h(x - x_n)^3)}{368640} \\ & + \frac{(73h^3(x - x_n))}{1290240} \end{aligned}$$

$$\begin{aligned} \beta_{\frac{2}{3}} = & \frac{-((3(x - x_n)^6))}{(50h^2)} - \frac{(9(x - x_n)^5)}{(400h)} - \frac{(38957h^2(x - x_n)^2)}{38707200} \\ & + \frac{(27(x - x_n)^8)}{(1400h^4)} + \frac{(5561h(x - x_n)^3)}{1843200} + \frac{(2053h^3(x - x_n))}{19353600} \end{aligned}$$

$$\begin{aligned}
& -\frac{(171(x-x_n)^7)}{(2800h^3)} \\
\beta_1 = & \frac{((17(x-x_n)^6))}{(2160h^2)} - \frac{(x-x_n)^5}{(360h)} - \frac{(62887h^2(x-x_n)^2)}{627056640} \\
& - \frac{(x-x_n)^7}{(112h^3)} + \frac{(x-x_n)^8}{(280h^4)} \frac{(215h(x-x_n)^3)}{663552} \\
& + \frac{(3119h^3(x-x_n))}{313528320}
\end{aligned}$$

Equation (4) is evaluated at the non-interpolating point  $x_{n+1}$  while its derivatives is evaluated at all points to produce the discrete schemes. The discrete schemes and its derivative are joined to form a block of the form

$$A^{[3]_4} Y_m = B^{[3]_4} R_1^{[3]_4} + h^4 \left[ D^{[3]_4} R_2^{[3]_4} + E^{[3]_4} R_3^{[3]_4} \right], \quad (8)$$

where  $A^{[3]_4}, B^{[3]_4}, R_1^{[3]_4}, D^{[3]_4}, R_2^{[3]_4}, E^{[3]_4}$  and  $R_3^{[3]_4}$  are shown in Appendix A.

Multiplying equation (8) by the inverse of  $A^{[3]_4}$ , we have

$$IY_m = \bar{B}_1^{[3]_4} R_1^{[3]_4} + h^4 \left[ \bar{D}^{[3]_4} R_2^{[3]_4} + \bar{E}^{[3]_4} R_3^{[3]_4} \right], \quad (9)$$

so  $\bar{B}_1^{[3]_4}, \bar{D}^{[3]_4}$  and  $\bar{E}^{[3]_4}$  are also clear in Appendix A

### 3. Analysis of the Method

#### 3.1. Order of the Method

The linear difference operator  $L$  associated with (9) is defined as

$$L[y(x); h] = \hat{I} \hat{Y}_m - \hat{\bar{B}}^{[3]_4} R_1^{[3]_4} - h^4 \left[ \hat{\bar{D}}^{[3]_4} R_2^{[3]_4} + \hat{\bar{E}}^{[3]_4} R_3^{[3]_4} \right] \quad (10)$$

where  $\hat{I}$  is identity matrix and  $\hat{\bar{B}}^{[3]_4}, \hat{\bar{D}}^{[3]_4}, \hat{\bar{E}}^{[3]_4}$  are  $4 \times 4$  coefficients matrices of the main block  $\hat{Y}_m = [y_{n+\frac{1}{4}}, y_{n+\frac{1}{3}}, y_{n+\frac{1}{4}}, y_{n+\frac{2}{3}}]^T$ . Expanding  $\hat{Y}_m$  and  $R_3^{[3]_4}$  components in Taylor's series respectively and collecting their terms in powers of  $h$  yields

$$L[y(x), h] = \bar{C}_0 y(x) + \bar{C}_1 h y'(x) + \bar{C}_2 h y''(x) + \dots \quad (11)$$

**Definition 1.** One step hybrid block method (9) and its linear operator (10) are said to have order  $p$ , if  $\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \dots = \bar{C}_{p+4} = 0$  and  $\bar{C}_{p+4} \neq 0$  with error constants vector  $\bar{C}_{p+4}$ .

Expanding (9) about  $x = x_n$  using Taylor series

$$\begin{aligned}
 & \left[ \sum_{j=0} \frac{(\frac{1}{4})^j h^j}{j!} y_n^j - y_n - \frac{h}{4} y_n' - \frac{h^2}{32} y_n'' - \frac{h^3}{384} y_n''' \right. \\
 & \quad - \frac{1525160066678784869h^4}{14708396095021850296320} y_n^{iv} \\
 & \quad - \frac{18463342850998261}{107727510461585817600} \sum_{j=0} \frac{(\frac{1}{4})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad + \frac{276458454908929}{2269814212194729984} \sum_{j=0} \frac{(\frac{1}{3})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad - \frac{234407082948819889}{21790216437069407846400} \sum_{j=0} \frac{(\frac{2}{3})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad \left. + \frac{-904854089194540327}{706003012561048814223360} \sum_{j=0} \frac{h^{j+4}}{j!} y_n^{j+4} \right] \\
 & \quad \sum_{j=0} \frac{(\frac{1}{3})^j h^j}{j!} y_n^j - y_n - \frac{h}{3} y_n' - \frac{h^2}{18} y_n'' - \frac{h^3}{162} y_n''' \\
 & \quad - \frac{157926226933125475h^4}{551564853563319386112} y_n^{iv} \\
 & \quad - \frac{491642959321279}{807956328461893632} \sum_{j=0} \frac{(\frac{1}{4})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad + \frac{31525197391593517}{76606229661572136960} \sum_{j=0} \frac{(\frac{1}{3})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad - \frac{432345564227566513}{12256996745851541913600} \sum_{j=0} \frac{(\frac{2}{3})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad + \frac{110488310858156393}{26475112971039330533376} \sum_{j=0} \frac{h^{j+4}}{j!} y_n^{j+4} \\
 & \quad \sum_{j=0} \frac{(\frac{2}{3})^j h^j}{j!} y_n^j - y_n - \frac{2h}{3} y_n' - \frac{2h^2}{9} y_n'' - \frac{4h^3}{81} y_n''' \\
 & \quad - \frac{2041631831074625743h^4}{689456066954149232640} y_n^{iv} \\
 & \quad - \frac{48038396025285287}{5049727052886835200} \sum_{j=0} \frac{(\frac{1}{4})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad + \frac{36028797018963977}{7660622966157213696} \sum_{j=0} \frac{(\frac{1}{3})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad - \frac{226466724119201927}{437749883780412211200} \sum_{j=0} \frac{(\frac{2}{3})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad + \frac{1921535841011412749}{33093891213799163166720} \sum_{j=0} \frac{h^{j+4}}{j!} y_n^{j+4} \\
 & \quad \sum_{j=0} \frac{h^j}{j!} y_n^j - y_n - h y_n' - \frac{h^2}{2} y_n'' - \frac{h^3}{6} y_n''' \\
 & \quad + \frac{904854089194540327h^4}{706003012561048814223360} y_n^{iv} \\
 & \quad - \frac{50158840849838897}{1346593880769822720} \sum_{j=0} \frac{(\frac{1}{4})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad + \frac{30399297484750853}{2837267765243412480} \sum_{j=0} \frac{(\frac{1}{3})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad - \frac{5836665117072161713}{1361888527316837990400} \sum_{j=0} \frac{(\frac{2}{3})^j h^{j+4}}{j!} y_n^{j+4} \\
 & \quad \left. + \frac{11673330234144328999}{44125188285065550888960} \sum_{j=0} \frac{h^{j+4}}{j!} y_n^{j+4} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
 \end{aligned}$$

Therefore,

$$\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_8 = 0,$$

and  $\bar{C}_{5+4} \neq 0$ .

Hence, the new method is of order  $[5, 5, 5, 5]^T$  with error constant

$$[1.667949e^{-9}, 5.369218e^{-9}, 7.168291e^{-8} - 2.362547e^{-6}]^T.$$

### 3.2. Zero Stability

One step hybrid block method (9) is said to be zero stable if the first characteristic polynomial  $\pi(z)$  having roots such that  $|z_r| \leq 1$ , and if  $|z_r| = 1$  then the multiplicity of  $z_r$  must not greater than two. Applying this in our method  $\hat{Y}_m$  where  $\hat{I}$  represents  $4 \times 4$  identity matrix and  $\hat{B}$  denotes the coefficients matrix of  $y_n$  yields

$$\begin{aligned} \Pi(r) &= |z \hat{I} - \hat{B}| \\ &= \left| z \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\ &= r^3(r - 1) \end{aligned}$$

whose solutions are  $r = 0, 0, 1, 1$ . Hence, the new main block method in (9) is zero stable. This implies, the new hybrid block method is convergent since it is zero stable and consistent ( $p \geq 1$ ), see Henrici, 1962.

### 3.3. Region of Absolute Stability

The locus method is employed to draw the region of absolute stability. Substituting the test equation  $y'''' = -\lambda^4 y$  in (9) where  $\bar{h} = \lambda^4 h^4$  and  $\lambda = \frac{df}{dy}$  gives characteristic function  $\Pi(r) = \rho(r) - \bar{h}\sigma(r)$ . let  $r = e^{i\theta} = \cos \theta - i \sin \theta$  and considering real part yields

$$\bar{h}(\theta, h) = \frac{2367600721920000(\cos(\theta) - 1)}{(3\cos(\theta) - 368)}, \quad (12)$$

which gives the region stability of the method as shown in Figure 1.

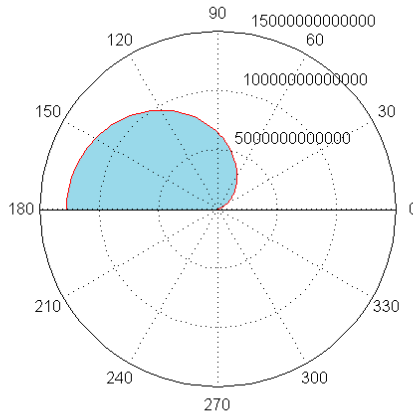


Figure 1: Region stability of new method

#### 4. Numerical Experiments

In this section, the performance of new one step hybrid block has been tested on one non-linear IVP and two linear IVPs. For each example, computed solution (COP), exact solution (EXT) and absolute errors (ERR) were determined using *Matlab* code, see Table 1 – Table 3.

##### Problem 1.

$$y^{iv} - (y')^2 + yy'' + 4x^2 - e^x(1 - 4x + x^2) = 0,$$

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 1, \quad h = \frac{1}{100}.$$

Exact solution:  $y(x) = x^2 + e^x$ .

##### Problem 2.

$$y^{iv} + y'' = 0, \quad y(0) = 0, \quad y'(0) = \frac{-1.1}{72 - 50\pi},$$

$$y'''(0) = \frac{1.2}{144 - 100\pi}, \quad h = \frac{1}{320}, \quad y''(0) = \frac{1}{144 - 100\pi}.$$

Exact solution:  $y(x) = \frac{1-x-\cos x-1.2\sin x}{144-100\pi}$ .



$x$		New method, $P = 5$	see [9], $P = 6$
0.003125	EXT	1.0031396535277390	1.003139653527739149
	CPS	1.0031396535277384	1.003139653526590265
	ERR	$6.661338e^{-16}$	$1.148884e^{-12}$
0.006250	EXT	1.0063086345037620	1.006308634503762010
	CPS	1.0063086345037602	1.006308634484910542
	ERR	$1.776357e^{-15}$	$1.8851468e^{-11}$
0.009375	EXT	1.0095069735890709	1.009506973589071086
	CPS	1.0095069735890683	1.009506973491318106
	ERR	$2.664535e^{-15}$	$9.7752980e^{-11}$
0.012500	EXT	1.0127347015406345	1.012734701540634377
	CPS	1.0127347015406303	1.012734701224875248
	ERR	$4.218847e^{-15}$	$3.15759129e^{-10}$
0.015625	EXT	1.0159918492116857	1.015991849211685747
	CPS	1.0159918492116808	1.015991848424806972
	Error	$4.884981e^{-15}$	$1.15463e^{-10}$

Table 1: Comparison of the new method with [9] for solving Problem 1, where  $h = \frac{1}{320}$

### Problem 3.

$$y^{iv} - x = 0, \quad y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0,$$

$$y'''(0) = 0, \quad h = \frac{1}{320}.$$

Exact solution:  $y(x) = \frac{x^5}{120} + x$ .

## 5. Conclusion

We have derived a new hybrid block method for the solution of linear and non-linear fourth order initial value problems. The obtained numerical results show the efficiency of the developed method in term of accuracy. Its performance is clear better than existing method through Table 1, Table 2 and Table 3. The new method developed is also found to be consistent, zero-stable, and then convergent.

$x$		New method, $P = 5$	[10], $P = 6$
0.103125	EXT	0.001300799589367158	0.00130079934027
	CPS	0.001300799589367138	0.00130079934027
	ERR	$1.929880e^{-17}$	$0.498732999343e^{-15}$
0.206250	EXT	0.002531773700195635	0.00253177321538
	CPS	0.00253177370019563	0.00253177321538
	ERR	$7.199102e^{-17}$	$0.676542155631e^{-15}$
0.306250	EXT	0.003652478978884993	0.00365247827946
	CPS	0.003652478978884832	0.00365247827947
	ERR	$1.608956e^{-16}$	$0.313507900196e^{-14}$
0.406250	EXT	0.004695953231804849	0.00469595233257
	CPS	0.004695953231804570	0.00469595233258
	ERR	$2.792905e^{-16}$	$0.943602834758e^{-14}$
0.506250	EXT	0.005657642360593461	0.00565764127720
	CPS	0.005657642360593029	0.00565764127722
	ERR	$4.310788e^{-16}$	$0.221168569570e^{-13}$

Table 2: Comparison of the new method with [10] for solving Problem 2, where  $h = \frac{1}{320}$ .

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$x$		New method, $P = 5$	[1], $P = 8$
0.1	EXT	0.100000083333333340	0.100000083333334000
	CPS	0.100000083333333340	0.10000008333351720
	ERR	$0.000000e^0$	$1.832e^{-13}$
0.2	EXT	0.200002666666666690	0.200002666666666900
	CPS	0.200002666666666580	0.20000266667150250
	ERR	$1.110223e^{-16}$	$4.835e^{-12}$
0.3	EXT	0.300020250000000040	0.300020250000000004
	CPS	0.300020249999999770	0.30002025000721480
	ERR	$2.775558e^{-16}$	$7.214e^{-12}$
0.4	EXT	0.400085333333333350	0.400008533333333333
	CPS	0.400085333333332740	0.40000853340160457
	ERR	$6.106227e^{-16}$	$6.832e^{-11}$
0.5	EXT	0.500260416666666650	0.50026041666666665
	CPS	0.500260416666665760	0.50026041674083458
	ERR	$8.881784e^{-16}$	$7.416e^{-11}$

Table 3: Comparison of the new method with [1] for solving Problem 3, where  $h = \frac{1}{10}$

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## 6. Appendix A

$$A^{[3]_4} = \begin{pmatrix} \frac{-128}{5} & 27 & \frac{-27}{5} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-128}{5} & \frac{18}{10h} & \frac{-9}{10h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5h}{52} & \frac{-45}{40h} & \frac{9}{40h} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5h}{64} & \frac{-12}{30} & \frac{-3}{69} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-128}{5} & \frac{30}{144} & \frac{10h}{-207} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-704}{5} & \frac{144}{-198} & \frac{10h}{63} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5h}{1152} & \frac{-198}{h^2} & \frac{5h^2}{-18} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5h^2}{288} & \frac{-36}{18} & \frac{5h^2}{-9} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{5h^2}{0} & \frac{18}{234} & \frac{-153}{5h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1152}{5h^2} & \frac{234}{450} & \frac{-261}{5h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-2304}{5h^2} & \frac{450}{648} & \frac{-324}{5h^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-3456}{5h^3} & \frac{648}{648} & \frac{-324}{5h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-3456}{5h^3} & \frac{648}{648} & \frac{-324}{5h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{-3456}{5h^3} & \frac{648}{648} & \frac{-324}{5h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{-3456}{5h^3} & \frac{648}{648} & \frac{-324}{5h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{-3456}{5h^3} & \frac{648}{648} & \frac{-324}{5h^3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$Y_m = \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{3}} \\ y_{n+\frac{2}{3}} \\ y_{n+1} \\ y'_{n+\frac{1}{4}} \\ y'_{n+\frac{1}{3}} \\ y'_{n+\frac{2}{3}} \\ y''_{n+1} \\ y''_{n+\frac{1}{3}} \\ y''_{n+\frac{2}{3}} \\ y'''_{n+1} \\ y'''_{n+\frac{1}{4}} \\ y'''_{n+\frac{1}{3}} \\ y'''_{n+\frac{2}{3}} \\ y_{n+1} \end{pmatrix},$$

$$B^{[3]_4} = \begin{pmatrix} -3 & 0 & 0 & 0 \\ \frac{-17}{(2h)} & -1 & 0 & 0 \\ \frac{-5}{(8h)} & 0 & 0 & 0 \\ \frac{1}{(2h)} & 0 & 0 & 0 \\ \frac{-5}{(2h)} & 0 & 0 & 0 \\ \frac{-35}{(2h)} & 0 & 0 & 0 \\ \frac{45}{h^2} & 0 & -1 & 0 \\ \frac{18}{h^2} & 0 & 0 & 0 \\ \frac{9}{h^2} & 0 & 0 & 0 \\ \frac{-27}{h^2} & 0 & 0 & 0 \\ \frac{-63}{h^2} & 0 & 0 & 0 \\ \frac{-108}{h^3} & 0 & 0 & -1 \\ \frac{-108}{h^3} & 0 & 0 & 0 \\ \frac{-108}{h^3} & 0 & 0 & 0 \\ \frac{-108}{h^3} & 0 & 0 & 0 \\ \frac{-108}{h^3} & 0 & 0 & 0 \end{pmatrix}, R_1^{[3]_4} = \begin{pmatrix} y_n \\ y_n' \\ y_n'' \\ y_n''' \\ y_n \end{pmatrix},$$

$$D^{[3]_4} = \begin{pmatrix} \frac{979h^4}{4976640} \\ \frac{-17329h^3}{104509440} \\ \frac{5935h^3}{668860416} \\ \frac{-5357h^3}{627056640} \\ \frac{7625h^3}{62705664} \\ \frac{82569326393009h^3}{78812993478983680} \\ \frac{2843665867144587h^2}{630503947831869440} \\ \frac{-17719067273067h^2}{78812993478983680} \\ \frac{-41h^2}{204120} \\ \frac{296061227673923h^2}{157625986957967360} \\ \frac{1832684344077h^2}{1231453023109120} \\ \frac{-1537285236573981h}{19703248369745920} \\ \frac{47401167953007h}{90071992547409920} \\ \frac{40668356010069h}{630503947831869440} \\ \frac{105105845795691h}{7881299347898368} \\ \frac{-3629h}{184320} \end{pmatrix},$$

$$E^{[3]4} = \begin{pmatrix} \frac{(-1213h^4)}{583200} & \frac{(137h^4)}{20480} & \frac{(1999h^4)}{921600} & \frac{(-461h^4)}{14929920} \\ \frac{(-24449h^3)}{12247200} & \frac{(-2548074565957h^3)}{45035996273704960} & \frac{(-2053h^3)}{19353600} & \frac{(3119h^3)}{313528320} \\ \frac{(12353h^3)}{78382080} & \frac{(167h^3)}{917504} & \frac{(253h^3)}{17694720} & \frac{(-2201h^3)}{2006581248} \\ \frac{(-7549h^3)}{73483200} & \frac{(-857h^3)}{3317760} & \frac{(-2033h^3)}{116121600} & \frac{(2419h^3)}{1881169920} \\ \frac{(-6887h^3)}{7348320} & \frac{(3233h^3)}{774144} & \frac{(2047h^3)}{3870720} & \frac{(-5911h^3)}{188116992} \\ \frac{(-1108604311169h^3)}{96207267430400} & \frac{(6739h^3)}{172032} & \frac{(14635578915999891h^3)}{788129934789836800} & \frac{(7336424722119h^3)}{39406496739491840} \\ \frac{(5128479470881h^2)}{131941395333120} & \frac{(-1507h^2)}{258048} & \frac{(50765836424615901h^2)}{25220157913274777600} & \frac{(-2023447286282923h^2)}{10088063165309911040} \\ \frac{(-468278332203551h^2)}{98516241848729600} & \frac{(-2895136283892861h^2)}{1261007895663738880} & \frac{(-3422820419919567h^2)}{12610078956637388800} & \frac{(112547357585937h^2)}{5044031582654955520} \\ \frac{(-352h^2)}{382725} & \frac{(-29h^2)}{3780} & \frac{(-19h^2)}{37800} & \frac{(11h^2)}{306180} \\ \frac{(-181382466546167h^2)}{9235897673318400} & \frac{(2398533712677243h^2)}{39406496739491840} & \frac{(4210783391832579h^2)}{394064967394918400} & \frac{(-520746202572989h^2)}{945755921747804160} \\ \frac{(-199588131815h^2)}{11544872091648} & \frac{(75119h^2)}{645120} & \frac{(418379808958713h^2)}{3518437208883200} & \frac{(3370668535027h^2)}{461794883665920} \\ \frac{(-7357h)}{21600} & \frac{(7513h)}{61440} & \frac{(-5561h)}{307200} & \frac{(215h)}{110592} \\ \frac{(79h)}{4320} & \frac{(-4853h)}{61440} & \frac{(-151h)}{61440} & \frac{(73h)}{552960} \\ \frac{(294205782948971h)}{4925812092436480} & \frac{(-443h)}{12288} & \frac{(-953h)}{307200} & \frac{(123018609613061h)}{630503947831869440} \\ \frac{(-529654659952219h)}{3078632557772800} & \frac{(7663550232614229h)}{19703248369745920} & \frac{(2633h)}{20480} & \frac{(-731428617866741h)}{157625986957967360} \\ \frac{(4931h)}{21600} & \frac{(-6311h)}{61440} & \frac{(93964355335705923h)}{197032483697459200} & \frac{(515840206952979h)}{4925812092436480} \end{pmatrix},$$

and

$$R_3^{[3]4} = \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{3}} \\ f_{n+\frac{2}{3}} \\ f_{n+1} \end{pmatrix}, \quad R_2^{[3]4} = \begin{pmatrix} f_n \end{pmatrix},$$

$$\bar{B}^{[3]4} = \begin{pmatrix} 1 & \frac{h}{4} & \frac{h^2}{3^2} & \frac{h^3}{384} \\ 1 & \frac{h}{3} & \frac{h^2}{18} & \frac{h^3}{162} \\ 1 & \frac{2h}{3} & \frac{2h^2}{9} & \frac{4h^3}{81} \\ 1 & h & \frac{h^2}{2} & \frac{h^3}{6} \\ 0 & 1 & \frac{h}{4} & \frac{h^2}{3^2} \\ 0 & 1 & \frac{h}{3} & \frac{h^2}{18} \\ 0 & 1 & \frac{2h}{3} & \frac{2h^2}{9} \\ 0 & 1 & h & \frac{h^2}{2} \\ 0 & 0 & 1 & \frac{h}{4} \\ 0 & 0 & 1 & \frac{h}{3} \\ 0 & 0 & 1 & \frac{2h}{3} \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \bar{D}^{[3]4} = \begin{pmatrix} 1525160066678784869h^4 \\ 14708396095021850296320 \\ 157926226933125475h^4 \\ 551564853563319386112 \\ 2041631831074625743h^4 \\ 689456066954149232640 \\ 1459166279268041291h^4 \\ 131324965134123663360 \\ 2730860328442135937h^3 \\ 1838549511877731287040 \\ 4120793659044005789h^3 \\ 1378912133908298465280 \\ 10079055966055174157h^3 \\ 689456066954149232640 \\ 3283124128353092033h^3 \\ 91927475593886564352 \\ 1862572697452545h^2 \\ 126100789566373888 \\ 279673536859707919h^2 \\ 13132496513412366336 \\ 31135997423796033h^2 \\ 630503947831869440 \\ 6755399441055747h^2 \\ 90071992547409920 \\ 49524935746038441h \\ 630503947831869440 \\ 7033399418053923h \\ 90071992547409920 \\ 514299957446631h \\ 5629499534213120 \\ 10344205394116613h \\ 177329235327713280 \end{pmatrix},$$

$$\bar{E}^{[3]4} = \begin{pmatrix} 18463342850998261h^4 & -276458454908929h^4 & 234407082948819889h^4 & -904854089194540327h^4 \\ 107727510461585817600 & 2269814212194729984 & 21790216437069407846400 & 706003012561048814223360 \\ 491642959321279h^4 & -31525197391593517h^4 & 432345564227566513h^4 & -110488310858156393h^4 \\ 807956328461893632 & 76606229661572136960 & 12256996745851541913600 & 26475112971039330533376 \\ 48038396025285287h^4 & -36028797018963977h^4 & 226466724119201927h^4 & -1921535841011412749h^4 \\ 5049727052886835200 & 76606229661572136960 & 437749883780412211200 & 33093891213799163166720 \\ 50158840849838897h^4 & -30399297484750853h^4 & 5836665117072161713h^4 & -11673330234144328999h^4 \\ 1346593880769822720 & 2837267765243412480 & 1361888527316837990400 & 44125188285065550888960 \\ 41479900791570421h^3 & -860387482870931h^3 & 498025990824393649h^3 & -1910880840330513703h^3 \\ 13465938807698227200 & 405323966463344640 & 2723777054633675980800 & 88250376570131101777920 \\ 79375943432404981h^3 & -128915539333480493h^3 & 864691128455134129h^3 & -3296634927235206439h^3 \\ 10099454105773670400 & 25535409887190712320 & 2042832790975256985600 & 66187782427598326333440 \\ 256705178760118261h^3 & -54043195528445957h^3 & 1128902306594203963h^3 & -11240984669916761383h^3 \\ 5049727052886835200 & 2837267765243412480 & 340472131829209497600 & 33093891213799163166720 \\ 410390516044136261h^3 & -45598946227126277h^3 & 17509995351216483281h^3 & -13132496513412497447h^3 \\ 3366484701924556800 & 2837267765243412480 & 680944263658418995200 & 22062594142532775444480 \\ 110461336172888059h^2 & -61344832346180833h^2 & 169571081282125711h^2 & -71670565944950881h^2 \\ 2659938529915699200 & 2269814212194729984 & 75660473739824332800 & 272377705463367598080 \\ 248260929458798581h^2 & -23h^2 & 2395915001761102769h^2 & -1297036692682703329h^2 \\ 3366484701924556800 & 540 & 680944263658418995200 & 3151799163218967920640 \\ 1681343860884985h^2 & -250199979298361h^2 & 4707762810477948923h^2 & -1345075088707988933h^2 \\ 9974769487183872 & 16888498602639360 & 226981421219472998400 & 817133116390102794240 \\ 11821949021847541h^2 & 0 & 10214163954876279913h^2 & 1513209474796482569h^2 \\ 41561539529932800 & -2061h & 75660473739824332800 & 272377705463367598080 \\ 323h & -19h & 801h & -167h \\ 900 & -120 & 51200 & 92160 \\ 266212777973455981h & 3152519739159347h & 3h & -29773797536504953h \\ 664984632478924800 & 11821949021847552 & 200 & 17023606591460474880 \\ 14011198840708199h & -9h & 11h & -28022397681416407h \\ 83123079059865600 & -40 & 75 & 4255901647865118720 \\ 128h & & 292593238290726857h & 13669128556511233h \\ 225 & & 591097451092377600 & 132996926495784960 \end{pmatrix}$$

